

CONNECTIONS, PATH LIFTINGS AND AMBROSE-SINGER THEOREM ON DIFFEOLOGICAL BUNDLES

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ABSTRACT. We develop here the notion of path-liftings, a generalization of horizontal lifts of paths to diffeological bundles for which the holonomy of a loop is well-defined. Then, we link it with the notion of connection on a diffeological bundle with regular Frölicher Lie group. After the statement of technical results and of a Lie theorem for subalgebras of Frölicher subalgebras, we state a reduction theorem and an Ambrose-Singer theorem for a class of diffeological bundles with regular Frölicher Lie groups. The case of (classical) principal bundles with structure group a Lie group G modeled on a locally convex complete topological vector space appears as a particular case.

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INTRODUCTION

This paper ends the study of the Ambrose-Singer like theorems that one can state out of the settings of finite dimensional manifolds. We began in [10] by stating an Ambrose-Singer theorem for a quite wide class of infinite dimensional principal bundles, addressing also open questions on the mathematical structure of the holonomy groups constructed. In this work, completed in [9], the most natural frameworks are not manifolds but weaker frameworks such as diffeological or Frölicher spaces. The starting point of [10] was a Lie theorem stated by Robart [14] for a wide class of Lie algebras, which helped us to circumvent the lack of easy-to-use Frobenius theorem on integrable distributions.

The approach is here slightly changed. The objects considered here are as general as possible: diffeological or Frölicher spaces. Infinite dimensional manifolds appear as particular cases. The notion of connection is constructed step by step, inspired by a generalization of Iglesias [5] of the notion of connection. This approach is adapted here to define path-liftings, a generalization of horizontal lifts of paths in section 2 which enables us to define holonomy groups. Connection forms are the infinitesimal aspect of G -invariant path-liftings. This leads us to an interesting classification of diffeological structure of a Frölicher space, a Lie theorem for Frölicher Lie subalgebras (Theorem 1.18) and the definition of regular Frölicher Lie groups with regular Lie algebra. These notions are applied to a very easy example: generalized Lie groups defined by Omori in [12]. All these results are technical tools which allows the machinery of [10] to work in this setting: we can build a second holonomy group, noted H^{red} , which is the smaller in a certain class to which we can apply the theorem of reduction of the structure group 4.2. We have in this setting the following Ambrose-Singer theorem: the Lie algebra of H^{red} is the regular Lie

algebra spanned by the curvature elements, that is the smaller regular Lie algebra which contains the curvature elements.

1. PRELIMINARIES ON DIFFERENTIABLE STRUCTURES

The objects of the category of -finite or infinite- dimensional smooth manifolds is made of topological spaces M equipped with a collection of charts called maximal atlas that enables one to make differentiable calculus. But in examples of projective limits of manifolds, a differential calculus is needed where as no atlas can be defined. To circumvent this problem which occurs in various frameworks, several authors have independently developed some ways to define differentiation without defining charts. We use here three of them. The first one is due to Souriau [13], the second one is due to Sikorski, and the third one is a setting closer to the setting of differentiable manifolds is due to Frölicher (see e.g. [1] for an introduction on these two last notions). In this section, we review some basics on these three notions.

1.1. Souriau's diffeological spaces, Sikorski's differentiable spaces, Frölicher spaces.

Definition 1.1. Let X be a set.

- A **plot** of dimension p (or p -plot) on X is a map from an open subset O of \mathbb{R}^p to X .
- A **diffeology** on X is a set \mathcal{P} of plots on X such that, for all $p \in \mathbb{N}$,
 - any constant map $\mathbb{R}^p \rightarrow X$ is in \mathcal{P} ;
 - Let I be an arbitrary set; let $\{f_i : O_i \rightarrow X\}_{i \in I}$ be a family of maps that extend to a map $f : \bigcup_{i \in I} O_i \rightarrow X$. If $\{f_i : O_i \rightarrow X\}_{i \in I} \subset \mathcal{P}$, then $f \in \mathcal{P}$.
 - (chain rule) Let $f \in \mathcal{P}$, defined on $O \subset \mathbb{R}^p$. Let $q \in \mathbb{N}$, O' an open subset of \mathbb{R}^q and g a smooth map (in the usual sense) from O' to O . Then, $f \circ g \in \mathcal{P}$.

- If \mathcal{P} is a diffeology on X , (X, \mathcal{P}) is called **diffeological space**.

Let (X, \mathcal{P}) et (X', \mathcal{P}') be two diffeological spaces, a map $f : X \rightarrow X'$ is **differentiable** (=smooth) if and only if $f \circ \mathcal{P} \subset \mathcal{P}'$.

Remark. Notice that any diffeological space (X, \mathcal{P}) can be endowed with the weaker topology such that all the maps that are in \mathcal{P} are continuous. But we prefer to mention this only for memory as well as other questions that are not closely related to our construction, and stay closer to the goals of this paper.

Let us now define the Sikorski's differential spaces. Let X be a Hausdorff topological space.

Definition 1.2. • A (Sikorski's) **differential space** is a pair (X, \mathcal{F}) where \mathcal{F} is a family of maps $X \rightarrow \mathbb{R}$ such that

- the topology of X is the initial topology with respect to \mathcal{F}
- for any $n \in \mathbb{N}$, for any smooth map $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$, for any $(f_1, \dots, f_n) \in \mathcal{F}^n$, $\varphi \circ (f_1, \dots, f_n) \in \mathcal{F}$.
- Let (X, \mathcal{F}) et (X', \mathcal{F}') be two differential spaces, a map $f : X \rightarrow X'$ is **differentiable** (=smooth) if and only if $\mathcal{F}' \circ f \subset \mathcal{F}$.

We now introduce Frölicher spaces.

Definition 1.3. • A **Frölicher space** is a triple $(X, \mathcal{F}, \mathcal{C})$ such that

- \mathcal{C} is a set of paths $\mathbb{R} \rightarrow X$,
- A function $f : X \rightarrow \mathbb{R}$ is in \mathcal{F} if and only if for any $c \in \mathcal{C}$, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$;

- A path $c : \mathbb{R} \rightarrow X$ is in \mathcal{C} (i.e. is a **contour**) if and only if for any $f \in \mathcal{F}$, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$.

• Let $(X, \mathcal{F}, \mathcal{C})$ et $(X', \mathcal{F}', \mathcal{C}')$ be two Frölicher spaces, a map $f : X \rightarrow X'$ is **differentiable** (=smooth) if and only if $\mathcal{F}' \circ f \circ \mathcal{C} \in C^\infty(\mathbb{R}, \mathbb{R})$.

Any family of maps \mathcal{F}_g from X to \mathbb{R} generate a Frölicher structure $(X, \mathcal{F}, \mathcal{C})$, setting [6]:

- $\mathcal{C} = \{c : \mathbb{R} \rightarrow X \text{ such that } \mathcal{F}_g \circ c \subset C^\infty(\mathbb{R}, \mathbb{R})\}$
- $\mathcal{F} = \{f : X \rightarrow \mathbb{R} \text{ such that } f \circ \mathcal{C} \subset C^\infty(\mathbb{R}, \mathbb{R})\}$.

One easily see that $\mathcal{F}_g \subset \mathcal{F}$. This notion will be useful in the sequel to describe in a simple way a Frölicher structure.

A Frölicher space, as a differential space, carries a natural topology, which is the pull-back topology of \mathbb{R} via \mathcal{F} . In the case of a finite dimensional differentiable manifold, the underlying topology of the Frölicher structure is the same as the manifold topology. In the infinite dimensional case, these two topologies differ very often.

In the three previous settings, we call X a **differentiable space**, omitting the structure considered. Notice that, in the three previous settings, the sets of differentiable maps between two differentiable spaces satisfy the chain rule. Let us now compare these three settings: One can see (see e.g. [1]) that we have the following, given at each step by forgetful functors:

$$\text{smooth manifold} \Rightarrow \text{Frölicher space} \Rightarrow \text{Sikorski differential space}$$

Moreover, one remarks easily from the definitions that, if f is a map from a Frölicher space X to a Frölicher space X' , f is smooth in the sense of Frölicher if and only if it is smooth in the sense of Sikorski.

One can remark, if X is a Frölicher space, we define a natural diffeology on X by [9]:

$$\mathcal{P}(\mathcal{F}) = \coprod_{p \in \mathbb{N}} \{f \text{ p-parametrization on } X; \mathcal{F} \circ f \in C^\infty(O, \mathbb{R}) \text{ (in the usual sense)}\}.$$

With this construction, we also get a natural diffeology when X is a Frölicher space. In this case, one can easily show the following:

Proposition 1.4. [9] *Let $(X, \mathcal{F}, \mathcal{C})$ and $(X', \mathcal{F}', \mathcal{C}')$ be two Frölicher spaces. A map $f : X \rightarrow X'$ is smooth in the sense of Frölicher if and only if it is smooth for the underlying diffeologies.*

Thus, we can also state:

$$\text{smooth manifold} \Rightarrow \text{Frölicher space} \Rightarrow \text{Diffeological space}$$

1.2. Push-forward, quotient and trace. We give here only the results that will be used in the sequel.

Proposition 1.5. [9] *Let (X, \mathcal{P}) be a diffeological space, and let X' be a set. Let $f : X \rightarrow X'$ be a surjective map. Then, the set*

$$f(\mathcal{P}) = \{u \text{ such that } u \text{ restricts to some maps of the type } f \circ p; p \in \mathcal{P}\}$$

*is a diffeology on X' , called the **push-forward diffeology** on X' by f .*

We have now the tools needed to describe the diffeology on a quotient:

Proposition 1.6. *let (X, \mathcal{P}) be a diffeological space and \mathcal{R} an equivalence relation on X . Then, there is a natural diffeology on X/\mathcal{R} , noted by \mathcal{P}/\mathcal{R} , defined as the push-forward diffeology on X/\mathcal{R} by the quotient projection $X \rightarrow X/\mathcal{R}$.*

Given a subset $X_0 \subset X$, where X is a Frölicher space or a diffeological space, we can define on trace structure on X_0 , induced by X .

- If X is equipped with a diffeology \mathcal{P} , we can define a diffeology \mathcal{P}_0 on X_0 setting

$$\mathcal{P}_0 = \{p \in \mathcal{P} \text{ such that the image of } p \text{ is a subset of } X_0\}.$$

- If $(X, \mathcal{F}, \mathcal{C})$ is a Frölicher space, we take as a generating set of maps \mathcal{F}_g on X_0 the restrictions of the maps $f \in \mathcal{F}$. In that case, the contours (resp. the induced diffeology) on X_0 are the contours (resp. the plots) on X which image is a subset of X_0 .

1.3. Cartesian products and projective limits. The category of Sikorski differential spaces is not cartesianly closed, see e.g. [1]. This is why we prefer to avoid the questions related to cartesian products on differential spaces in this text, and focus on Frölicher and diffeological spaces, since the cartesian product is a tool essential for the definition of configuration spaces.

In the case of diffeological spaces, we have the following [13]:

Proposition 1.7. *Let (X, \mathcal{P}) and (X', \mathcal{P}') be two diffeological spaces. We call **product diffeology** on $X \times X'$ the diffeology $\mathcal{P} \times \mathcal{P}'$ made of plots $g : O \rightarrow X \times X'$ that decompose as $g = f \times f'$, where $f : O \rightarrow X \in \mathcal{P}$ and $f' : O \rightarrow X' \in \mathcal{P}'$.*

Then, in the case of a Frölicher space, we derive very easily, compare with e.g. [6]:

Proposition 1.8. *Let $(X, \mathcal{F}, \mathcal{C})$ and $(X', \mathcal{F}', \mathcal{C}')$ be two Frölicher spaces, with natural diffeologies \mathcal{P} and \mathcal{P}' . There is a natural structure of Frölicher space on $X \times X'$ which contours $\mathcal{C} \times \mathcal{C}'$ are the 1-plots of $\mathcal{P} \times \mathcal{P}'$.*

We can even state the same results in the case of infinite products, but it is useless here. More useful is the description of what happens for projective limits of Frölicher and diffeological spaces.

Proposition 1.9. *Let Λ be an infinite set of indexes, that can be uncountable.*

- *Let $\{(X_\alpha, \mathcal{P}_\alpha)\}_{\alpha \in \Lambda}$ be a family of diffeological spaces indexed by Λ totally ordered for inclusion, as in section 5, with $(i_{\beta, \alpha})_{(\alpha, \beta) \in \Lambda^2}$ a family of diffeological maps. If $X = \bigcap_{\alpha \in \Lambda} X_\alpha$, X carries the **projective diffeology** \mathcal{P} which is the pull-back of the diffeologies \mathcal{P}_α of each X_α via the family of maps $(f_\alpha)_{\alpha \in \Lambda}$. The diffeology \mathcal{P} made of plots $g : O \rightarrow X$ such that, for each $\alpha \in \Lambda$,*

$$f_\alpha \circ g \in \mathcal{P}_\alpha.$$

This is the biggest diffeology for which the maps f_α are smooth.

- *Let $\{(X_\alpha, \mathcal{F}_\alpha, \mathcal{C}_\alpha)\}_{\alpha \in \Lambda}$ be a family of Frölicher spaces indexed by Λ totally ordered for inclusion, as in section 5, with $(i_{\beta, \alpha})_{(\alpha, \beta) \in \Lambda^2}$ a family of differentiable maps. with natural diffeologies \mathcal{P}_α . There is a natural structure of Frölicher space $X = \bigcap_{\alpha \in \Lambda} X_\alpha$, which contours*

$$\mathcal{C} = \bigcap_{\alpha \in \Lambda} \mathcal{C}_\alpha$$

are the 1-plots of $\mathcal{P} = \bigcap_{\alpha \in \Lambda} \mathcal{P}_\alpha$. A generating set of functions for this Frölicher space is the set of maps of the type:

$$\bigcup_{\alpha \in \Lambda} \mathcal{F}_\alpha \circ f_\alpha.$$

1.4. Differential forms on a diffeological space and differential dimension.

Definition 1.10. [13] Let (X, \mathcal{P}) be a diffeological space and let V be a vector space equipped with a differentiable structure. A V -valued n -differential form α on X (noted $\alpha \in \Omega^n(X, V)$) is a map

$$\alpha : \{p : O_p \rightarrow X\} \in \mathcal{P} \mapsto \alpha_p \in \Omega^n(p; V)$$

such that

- Let $x \in X$. $\forall p, p' \in \mathcal{P}$ such that $x \in \text{Im}(p) \cap \text{Im}(p')$, the forms α_p and $\alpha_{p'}$ are of the same order n .

- Moreover, let $y \in O_p$ and $y' \in O_{p'}$. If (X_1, \dots, X_n) are n germs of paths in $\text{Im}(p) \cap \text{Im}(p')$, if there exists two systems of n -vectors $(Y_1, \dots, Y_n) \in (T_y O_p)^n$ and $(Y'_1, \dots, Y'_n) \in (T_{y'} O_{p'})^n$, if $p_*(Y_1, \dots, Y_n) = p'_*(Y'_1, \dots, Y'_n) = (X_1, \dots, X_n)$,

$$\alpha_p(Y_1, \dots, Y_n) = \alpha_{p'}(Y'_1, \dots, Y'_n).$$

We note by

$$\Omega(X; V) = \bigoplus_{n \in \mathbb{N}} \Omega^n(X, V)$$

the set of V -valued differential forms.

With such a definition, we feel the need to make two remarks for the reader:

- If there does not exist n linearly independent vectors (Y_1, \dots, Y_n) defined as in the last point of the definition, $\alpha_p = 0$ at y .
- Let $(\alpha, p, p') \in \Omega(X, V) \times \mathcal{P}^2$. If there exists $g \in C^\infty(D(p); D(p'))$ (in the usual sense) such that $p' \circ g = p$, then $\alpha_p = g^* \alpha_{p'}$.

Proposition 1.11. The set $\mathcal{P}(\Omega^n(X, V))$ made of maps $q : x \mapsto \alpha(x)$ from an open subset O_q of a finite dimensional vector space to $\Omega^n(X, V)$ such that for each $p \in \mathcal{P}$,

$$\{x \mapsto \alpha_p(x)\} \in C^\infty(O_q, \Omega^n(O_p, V)),$$

is a diffeology on $\Omega^n(X, V)$.

Working on plots of the diffeology, one can define the product and the differential of differential forms, which have the same properties as the product and the differential of differential forms.

Definition 1.12. Let (X, \mathcal{P}) be a diffeological space.

- (X, \mathcal{P}) is **finite-dimensional** if and only if

$$\exists n_0 \in \mathbb{N}, \quad \forall n \in \mathbb{N}, \quad n \geq n_0 \Rightarrow \dim(\Omega^n(X, \mathbb{R})) = 0.$$

Then, we set

$$\dim(X, \mathcal{P}) = \max\{n \in \mathbb{N} | \dim(\Omega^n(X, \mathbb{R})) > 0\}.$$

- If not, (X, \mathcal{P}) is called **infinite dimensional**.

Let us make a few remarks on this definition. If X is a manifold with $\dim(X) = n$, the natural diffeology as described in section 1.1 (also called “nébuleuse” diffeology) is such that

$$\dim(X, \mathcal{P}_0) = n.$$

Now, if $(X, \mathcal{F}, \mathcal{C})$ is the natural Frölicher structure on X , take \mathcal{P}_1 generated by the maps of the type $g \circ c$, where $c \in \mathcal{C}$ and g is a smooth map from an open subset of a finite dimensional space to \mathbb{R} . This is an easy exercise to show that

$$\dim(X, \mathcal{P}_1) = 1.$$

This first point shows that the dimension depends on the diffeology considered. Now, we remark that \mathcal{F} is the set of smooth maps $(X, \mathcal{P}_1) \rightarrow \mathbb{R}$,

This leads to the following definition, since $\mathcal{P}(\mathcal{F})$ is clearly the diffeology with the biggest dimension associated to $(X, \mathcal{F}, \mathcal{C})$:

Definition 1.13. The **dimension** of a Frölicher space $(X, \mathcal{F}, \mathcal{C})$ is the dimension of the diffeological space $(X, \mathcal{P}(\mathcal{F}))$.

1.5. Regular Frölicher groups. Let $(G, \mathcal{F}, \mathcal{C})$ be a Frölicher space which is a group such that the group law and the inversion map are smooth. These laws are also smooth for the underlying diffeology. Then, following [7], this is possible as in the case of manifolds to define a tangent space and a Lie algebra \mathfrak{g} of G using germs of smooth maps. Let us precise the algebraic, diffeological and Frölicher structures of \mathfrak{g} .

Proposition 1.14. Let $\mathfrak{g} = \{\partial_t c(0); c \in \mathcal{C} \text{ and } c(0) = e_G\}$ be the space of germs of paths at e_G .

- Let $(X, Y) \in \mathfrak{g}^2$, $X + Y = \partial_t(c.d)(0)$ where $c, d \in \mathcal{C}$, $c(0) = d(0) = e_G$, $X = \partial_t c(0)$ and $Y = \partial_t d(0)$.
- Let $(X, g) \in \mathfrak{g} \times G$, $Ad_g(X) = \partial_t(gcg^{-1})(0)$ where $c \in \mathcal{C}$, $c(0) = e_G$, and $X = \partial_t c(0)$.
- Let $(X, Y) \in \mathfrak{g} \times G$, $[X, Y] = \partial_t(Ad_{c(t)}Y)$ where $c \in \mathcal{C}$, $c(0) = e_G$, $X = \partial_t c(0)$.

All these operations are smooth and thus well-defined.

The basic properties remain globally the same as in the case of Lie groups, and the proofs are similar replacing charts by plots of the underlying diffeologies. (see e.g. [7] for further details)

Definition 1.15. A Frölicher group G with Lie algebra \mathfrak{g} is called **regular** if and only if there is a smooth map

$$Exp : C^\infty([0; 1], \mathfrak{g}) \rightarrow C^\infty([0, 1], G)$$

such that $g(t) = Exp(v(t))$ if and only if g is the unique solution of the differential equation

$$\begin{cases} g(0) = e \\ \frac{dg(t)}{dt} g(t)^{-1} = v(t) \end{cases}$$

We define

$$\begin{aligned} exp : \mathfrak{g} &\rightarrow G \\ v &\mapsto exp(v) = g(1) \end{aligned}$$

where g is the image by Exp of the constant path v .

The classical setting for infinite dimensional differential geometry requires the model topological vector space to be complete or Mac-Key complete. One of the reasons for this choice is to ensure the existence of the integral of a path over a compact interval. This means that the choice of an adequate topology is necessary.

For vector spaces, the basis for such a study can be found in [6], when the properties of the so-called “convenient vector spaces” are given. We have to remark that a vector space for which addition and scalar multiplication are compatible with a given Frölicher structure needs only a topological structure to become a convenient vector space. In order to circumvent these topological considerations, and adapting the terminology of regular Lie groups to vector spaces (which are viewed as abelian Lie groups), we set:

Definition 1.16. Let $(V, \mathcal{F}, \mathcal{C})$ be a Frölicher vector space, i.e. a vector space V equipped with a Frölicher structure compatible with the vector space addition and the scalar multiplication. $(V, \mathcal{F}, \mathcal{C})$ is **regular** if there is a smooth map

$$\int_0^{(\cdot)} : C^\infty([0; 1]; V) \rightarrow C^\infty([0; 1], V)$$

such that $\int_0^{(\cdot)} v = u$ if and only if u is the unique solution of the differential equation

$$\begin{cases} u(0) = 0 \\ u'(t) = v(t) \end{cases}.$$

This definition is of course fulfilled if V is a complete locally convex topological vector space, equipped with its natural Frölicher structure.

Definition 1.17. Let G be a Frölicher Lie group with Lie algebra \mathfrak{g} . Then, G is **regular with regular Lie algebra** if both G and \mathfrak{g} are regular.

Theorem 1.18. Let G be a regular Frölicher Lie group with Lie algebra \mathfrak{g} . Let \mathfrak{g}_1 be a Lie subalgebra of \mathfrak{g} . Let $G_1 = \text{Exp}(C^\infty([0; 1]; \mathfrak{g}_1))(1)$. If $\text{Ad}_{G_1 \cup G_1^{-1}}(\mathfrak{g}_1) = \mathfrak{g}_1$, G_1 is a Frölicher subgroup of G .

Proof. G_1 is obviously a Frölicher subspace of G . All we need to show is that G_1 is a subgroup of G (algebraically). This is a well-known procedure, found by Robart [14] to our knowledge:

- If $g = \text{Exp}(u(\cdot))(1) \in G_1$ and $g' = \text{Exp}(v(\cdot))(1) \in G_1$, with u and v smooth paths that are stationary at the endpoints, $gg' = \text{Exp}(\text{Ad}_g(v) \vee u) \in G_1$.
- If $g = \text{Exp}(u(\cdot))(1)$ and $g(t) = \text{Exp}(u(\cdot))(t)$, $g^{-1} = \text{Exp}(-\text{Ad}_{g^{-1}}u(\cdot))(1) \in G_1$. \square

We now turn to Omori’s generalized Lie groups.

Proposition 1.19. Let $(G_n)_{n \in \mathbb{N}}$ be a sequence of Banach Lie groups, increasing for \supset , and such that the inclusions are Lie group morphisms. Let $G = \bigcap_{n \in \mathbb{N}} G_n$. Then, G is a Frölicher regular Lie group with regular Lie algebra $\mathfrak{g} = \bigcap_{n \in \mathbb{N}} \mathfrak{g}_n$.

Proof. In this proof, each Lie group G_n is equipped with its natural Frölicher structure of smooth manifold $(G_n, \mathcal{F}_n, \mathcal{C}_n)$, with underlying diffeology \mathcal{P}_n . The group G is equipped with the projective Frölicher structure $(G, \mathcal{F}, \mathcal{C})$, with underlying diffeology \mathcal{P} . Let $f, g \in \mathcal{P}^2$ and $\alpha \in \Lambda$. Then, $f_\alpha \circ (f \cdot g) = (f_\alpha \circ f) \cdot (f_\alpha \circ g) \in \mathcal{P}_\alpha$ and $f_\alpha \circ g^{-1} = (f_\alpha \circ g)^{-1} \in \mathcal{P}_\alpha$. Thus, multiplication and inversion are differentiable in G , in the sense of diffeologies and hence in the sense of Frölicher. G is a Frölicher Lie group.

We now look at \mathfrak{g} , which equals to $\bigcap_{n \in \mathbb{N}} \mathfrak{g}_n$ trivially. Then, for each $n \in \mathbb{N}$, if Exp_n is the exponential on G_n , $\forall m < n$, $\text{Exp}_m \circ D_e i_{m,n} = \text{Exp}_n$ since $i_{m,n}$ is a morphism of Lie groups. (in fact, if we want to be rigorous, we need to replace $D_e i_{m,n}$ by the

map $C^\infty([0; 1]; \mathfrak{g}_n) \rightarrow C^\infty([0; 1]; \mathfrak{g}_m)$ induced by $i_{m,n}$. Thus, the exponential on G is smooth. \square

2. PATH LIFTINGS

2.1. The general setting for path-lifting. Let X and M be two differentiable spaces, and $\pi : X \rightarrow M$ a smooth surjective map. If γ is a path, we set $\gamma^{-1}(t) = \gamma(1 - t)$. We define $\mathcal{B}(X; M)$ the set of couples of the type (γ, x) such that γ is a smooth path on M and $x \in \pi^{-1}(\gamma(0))$. It is a natural (trace) differentiable space.

Definition 2.1. , A **path-lifting** $L : (\gamma; x) \mapsto L_x(\gamma)$ is a smooth map from $\mathcal{B}(X; M)$ to the set of smooth paths on X , which satisfy the following properties :

- (i) $\forall \gamma \in C^\infty([0, 1], M), \forall x \in \pi^{-1}(\gamma(0)),$

$$\pi \circ (L_x(\gamma)) = \gamma;$$
- (ii) $\forall \gamma \in C^\infty([0, 1], M)$ such that $\gamma = \gamma' \vee \gamma'', \forall x \in \pi^{-1}(\gamma(0)),$

$$L_x(\gamma) = L_{L_x(\gamma'')(1)}(\gamma') \vee L_x(\gamma''),$$
- (iii) $\forall \gamma \in C^\infty([0, 1], M), \forall g \in C^\infty([0, 1], [0, 1])$ such that g is monotone, $\forall x \in \pi^{-1}(\gamma(0)),$

$$L_x(\gamma \circ g) = L_x(\gamma) \circ g.$$
- (iv) $\forall (\gamma, x) \in \mathcal{B}(X, M),$

$$L_x(\gamma^{-1} \vee \gamma)(1) = L_x(\gamma)(0) = x.$$
- (v) $\forall (\gamma, x) \in \mathcal{B}(X, M), L_x(\gamma)(0) = x.$
- (vi) Let $\gamma \in C^\infty([0; 1]; M).$

$$(\exists x \in \pi^{-1}(\gamma(0)), L_x(\gamma)(1) = x) \Leftrightarrow (\forall x \in \pi^{-1}(\gamma(0)), L_x(\gamma)(1) = x)$$

One can recognize here all the basic key properties of the horizontal lift of a connection on a fiber bundle when the fiber is a compact manifold without boundary [11]. We need to precise an ambiguity due to the “smooth” setting: the product of paths is often developed for **continuous** (and not smooth) paths, because the condition $\gamma(1) = \gamma'(0)$ is not sufficient to get a smooth path $\gamma' \vee \gamma$ if γ and γ' are smooth. This is why we need to reparametrize paths into paths which are stationary at each endpoint. This does not change anything for path lifting (except the parametrization), because of property (iii). For the details in a context of connections on principal bundles, see e.g. [9].

We now have to extend the classical construction of the holonomy group of a connection to the context of path liftings.

Let $x \in X, m = \pi(x)$. We set

$$\mathcal{L}(m, M) = \{ \gamma : [0; 1] \rightarrow M \text{ such that } \gamma \text{ is a loop based at } m, \text{ stationary at endpoints} \}$$

Let $\mathcal{L}_0^L(m; M) = \{ \gamma \in \mathcal{L}(m; M) \text{ such that } L_x(\gamma)(1) = x \}$. Notice that all the loops of the type $\gamma^{-1} \vee \gamma$ are in $\mathcal{L}_0^L(m; M)$ by (iv) of Definition 2.1.

Definition 2.2. For $\gamma, \gamma' \in \mathcal{L}(m; M),$

$$\gamma \sim \gamma' \Leftrightarrow \exists x \in \pi^{-1}(m), L_x(\gamma)(1) = L_x(\gamma')(1)$$

Using (ii) of Definition 2.1, we get immediately:

Proposition 2.3.

$$\gamma \sim \gamma' \Leftrightarrow (\gamma')^{-1} \vee \gamma \in \mathcal{L}_0^L(m; M) .$$

Using (vi) of Definition 2.1, we see:

Proposition 2.4.

$$\gamma \sim \gamma' \Leftrightarrow \forall x \in \pi^{-1}(m), L_x(\gamma)(1) = L_x(\gamma')(1) .$$

Finally, by (iii) of Definition 2.1, we get the last easy property:

Proposition 2.5. *if $\tilde{\gamma}$ is a reparametrization of $\gamma \in \mathcal{L}(m; M)$, $\tilde{\gamma} \sim \gamma$.*

Noticing that this relation is symmetric and transitive, we set

$$\mathcal{H}_x^L = \mathcal{L}(m; M) / \sim .$$

• *push-forward of \vee and of the inversion of paths*

First, we notice that,

- if $\gamma \sim \gamma'$, $L_{L_x(\gamma)(1)}(\gamma^{-1})(1) = x = L_{L_x(\gamma')(1)}(\gamma'^{-1})(1)$ hence $\gamma^{-1} \sim \gamma'^{-1}$.
- if $\gamma \sim \gamma'$ and $\delta \sim \delta'$, $\forall x \in \pi^{-1}(m)$,

$$L_x(\gamma \vee \delta)(1) = L_{L_x(\delta)(1)}(\gamma)(1) = L_{L_x(\delta')(1)}(\gamma)(1) = L_{L_x(\delta')(1)}(\gamma')(1) = L_x(\gamma' \vee \delta')(1).$$

Hence $\gamma \vee \delta \sim \gamma' \vee \delta'$.

• *Associativity.* Given three loops γ , γ' and γ'' , $\gamma \vee (\gamma' \vee \gamma'')$ and $(\gamma \vee \gamma') \vee \gamma''$ differ by parametrizations. Hence, $\gamma \vee (\gamma' \vee \gamma'') \sim (\gamma \vee \gamma') \vee \gamma''$.

We note also by \vee and $^{-1}$ the push forward of \vee and $^{-1}$ onto \mathcal{H}_x^L . We can now state, by push-forward of the differentiable structure of $L(M, m)$:

Theorem 2.6. *(\mathcal{H}_x^L, \vee) is a diffeological group, and the inversion $^{-1}$ is differentiable.*

2.2. Comments and remarks. • Let $\gamma \in \mathcal{L}(m; M)$. The map $L_{(\cdot)}(\gamma)$ is a smooth map on $\pi^{-1}(m)$, with smooth inverse $L_{(\cdot)}(\gamma^{-1})$. Then, for fixed $m \in M$, L defines a map $\mathcal{L}(m; M) \rightarrow \text{Diff}(\pi^{-1}(m))$, where $\text{Diff}(\pi^{-1}(m))$ is the group of (diffeological) diffeomorphisms of the fiber. Moreover,

Proposition 2.7. *Let $m \in M$ and $\gamma \in \mathcal{L}(m; M)$.*

$$L_{(\cdot)}(\gamma' \vee \gamma) = L_{(\cdot)}(\gamma') \circ L_{(\cdot)}(\gamma),$$

Proof. Straightforward from $\forall \gamma \in C^\infty([0, 1], M)$ such that $\gamma = \gamma' \vee \gamma''$, $\forall x \in \pi^{-1}(\gamma(0))$,

$$L_x(\gamma) = L_{L_x(\gamma'')(1)}(\gamma') \vee L_x(\gamma'') \quad \square$$

Proposition 2.8. *Let $m \in M$ and $\gamma \in \mathcal{L}(m; M)$.*

$$L_{(\cdot)}(\gamma) = \text{Id}_{\pi^{-1}(m)} \Leftrightarrow \gamma \in \mathcal{L}_0^L(m; M)$$

Proof. Straightforward from: Let $\gamma \in C^\infty([0, 1]; M)$.

$(\exists x \in \pi^{-1}(\gamma(0)))$ such that $L_x(\gamma)(1) = x \Leftrightarrow (\forall x \in \pi^{-1}(\gamma(0)))$ such that $L_x(\gamma)(1) = x \quad \square$

Then, we have:

Theorem 2.9. *Let $m \in M$ and $x \in \pi^{-1}(M)$. The map L induces a quotient map*

$$\mathcal{H}_x^L \rightarrow \text{Diff}(\pi^{-1}(m)).$$

- Let (P, M, G) be a (classical) finite dimensional principal bundle of basis M and of structure group G . Any connection on P induces a path-lifting. It seems that the inverse induction is not elementary, and is not true in general, because the structure group G (which models the fiber) is viewed here only as a manifold, without any group structure. Maybe a stronger analysis could give more details on the correspondence, for example up to homotopy, between general path liftings and path liftings induced by connections.

- Let (P, M, N) be a finite dimensional fiber bundle of basis M and of typical fiber N . Then, if $\text{Diff}(N)$ is a Lie group, $P \times_N \text{Diff}(N)$ is a principal bundle and there is a bijection between fiber bundle connections on (P, M, N) and (classical) connections on $P \times_N \text{Diff}(N)$. This bijection is established in e.g. [6] or [11]. But, if N is not compact, horizontal lifts of paths are not well-defined for an arbitrary connection, and one has problems to define a holonomy group where as curvature elements are well-defined. This comes from the definition of horizontal lifts, which are defined as solutions of a differential equation: $\tilde{\gamma} \in C^\infty([0; 1]; P)$ is a horizontal lift of $\gamma \in C^\infty([0; 1]; M)$ if $\pi(\tilde{\gamma}) = \gamma$ and if $D\tilde{\gamma}$ is horizontal. With such a definition, it is obvious that two horizontal lifts of a same path γ differ by their starting point, but also that all starting points are not good to define horizontal lifts for γ when N is not compact. With our setting, such problems are avoided since only “good” connections, that is connections for which horizontal lifts exist, are considered.

- When X , a diffeological space, is equipped with a relation of equivalence \mathcal{R} , the quotient space $M = X/\mathcal{R}$ (with quotient projection $\pi : X \rightarrow M$) is also a diffeological space by push-forward of the diffeology of X . Then, our notion of path-lifting is also valid here, as well as holonomy. In a future work, we shall precise the role of the condition (vi) on the existence of an isomorphism between two equivalence classes $\pi^{-1}(m)$ and $\pi^{-1}(m')$ for $(m, m') \in M^2$. A counterexample to invariance of the holonomy group will be developped elsewhere, in the framework of infinite configuration spaces.

2.3. Homotopy, fundamental group and holonomy of a path-lifting. As mentioned in [13], the notion of homotopy of paths can be adapted straightway from the category of topological spaces to the category of diffeological spaces. These two notions coincide on the subcategory of smooth finite dimensional (paracompact) manifolds, since smooth or continuous homotopy gives the same equivalence classes of maps in this restricted class of objects. We also recall that one can adapt straightway the definition of arcwise connected components to the setting of diffeological spaces:

Definition 2.10. Let (X, \mathcal{P}) be a diffeological group. Let $(x, y) \in X^2$. x and y are in the same (arcwise) connected component if there is a smooth path γ on X starting from x and ending on y .

Let us now recall the definitions and the key properties of the homotopy of loops in the context of diffeological spaces:

Let X be a diffeological space. Let γ and γ' be two smooth loops on X , based on x .

- A **homotopy** between γ and γ' is a smooth map

$$H : S^1 \times [0; 1] \rightarrow X$$

such that $H(1, \cdot) = x$, $H(\cdot, 0) = \gamma$ and $H(\cdot, 1) = \gamma'$.

- γ and γ' are called **homotopic** if there exists a homotopy between γ and γ' .
- $\pi_1(X, x)$ is the set of connected components of smooth loops. This is the space of equivalence classes of loops modulo homotopy. If we consider only loops constant at endpoints, $\pi_1(X, x)$ gets a group structure induced by the composition of paths \vee .
- $\pi_1(X, x)$ can be identified to the set of connected components of $\mathcal{L}(X, x)$.

From this last property, with the notations of section 2.1, any path lifting L induces a (onto) map from $\pi_1(M, m)$ to the connected components of \mathcal{H}_x^L .

Definition 2.11. A path lifting is **flat** at $x \in X$ if the connected components of \mathcal{H}_x^L are made of singletons.

A path lifting in **totally flat** at $x \in X$ if the set \mathcal{H}_x^L has only one element.

With this definition, we get the following obvious statement:

Proposition 2.12. *If L is flat at x , there is a surjective group morphism*

$$\pi_1(M, m) \rightarrow \mathcal{H}_x^L$$

induced by the path lifting

$$L : \mathcal{L}(M, m) \rightarrow \text{paths starting at } x.$$

As a consequence, if there exists a totally flat path lifting, M is simply connected at m .

3. DIFFEOLOGICAL PRINCIPAL BUNDLES WITH REGULAR (FRÖLICHER) GROUPS

Let P be a diffeological space and let G be a regular Frölicher Lie group, with a differentiable right-action $P \times G \rightarrow P$, such that $\forall (p, p', g) \in P \times P \times G$, we have $p \cdot g = p' \cdot g \Rightarrow p = p'$. Let $M = P/G$, equipped with the quotient diffeology.

Proposition 3.1. *Let V be a vector space. G acts smoothly on the right on $\Omega(P, V)$ setting*

$$\forall (g, \alpha) \in \Omega^n(P, V) \times G, \forall p \in \mathcal{P}(P), (g_* \alpha)_{g \cdot p} = \alpha_p \circ (dg^{-1})^n.$$

Proof. G acts smoothly on P so that, if $p \in \mathcal{P}(P)$, $g \cdot p \in \mathcal{P}(P)$. The right action is now well-defined, and the smoothness is trivial.

Definition 3.2. Let $\alpha \in \Omega(P; \mathfrak{g})$. The differential form α is **right-invariant** if and only if, for each $p \in \mathcal{P}(P)$, and for each $g \in G$,

$$\alpha_{g \cdot p} = Ad_{g^{-1}} \circ g_* \alpha_p.$$

Now, let us turn to holonomy. Let $p \in P$ and γ a smooth path in P starting at p . We can now turn to connexions and their holonomy:

Definition 3.3. A **connection** on P is a \mathfrak{g} -valued 1-form θ , right-invariant, such that, for each $g \in \mathfrak{g}$, for any path $c : \mathbb{R} \rightarrow G$ such that $\begin{cases} c(0) &= e_G \\ c'(0) &= g \end{cases}$, and for each $p \in P$,

$$\theta((p \cdot c(t))'_{t=0}) = g.$$

Now, let us turn to holonomy. Let $p \in P$ and γ a smooth path in P starting at p , defined on $[0; 1]$. Let $H\gamma(t) = \gamma(t)g(t)$ where $g(t) \in C^\infty([0; 1]; \mathfrak{g})$ is a path satisfying the differential equation:

$$\begin{cases} \theta((H\gamma(t))') = 0 \\ H\gamma(0) = \gamma(0) \end{cases}$$

The first line of the definition is equivalent to the differential equation $g^{-1}(t)(g(t))' = -\theta(\gamma'(t))$, and the second to $g(0) = e_G$, which is integrable. This shows that horizontal lifts are well-defined, as in the case of manifolds. Moreover, the map $H(\cdot)$ defines trivially a path-lifting. This enables us to consider the holonomy group of the connection. Notice that a straightforward adaptation of the arguments of [9] shows that the holonomy group is invariant (up to conjugation) under the choice of the basepoint p . We note it now \mathcal{H} , omitting the basepoint p and the connection θ in our notations since they are assumed fixed.

4. CURVATURE AND THE LIE ALGEBRA OF THE HOLONOMY GROUP

Now, we assume that $\dim(P) \geq 2$. We fix a connection θ on P .

Definition 4.1. Let $\alpha \in \Omega(P; \mathfrak{g})$ be a G -invariant form. Let $\nabla\alpha = d\alpha - \frac{1}{2}[\theta, \alpha]$ be the horizontal derivative of α . We set

$$\Omega = \nabla\theta$$

the curvature of θ .

4.1. Reduction of the structure group. We now turn to reduction of the structure group, adapting a theorem from [10]:

Theorem 4.2. *We assume that G_1 and G are regular Frölicher groups with regular Lie algebras \mathfrak{g}_1 and \mathfrak{g} . Let $\rho : G_1 \mapsto G$ be an injective morphism of Lie groups. If there exists a connection θ on P , with curvature Ω , such that, for any smooth 1-parameter family Hc_t of horizontal paths starting at p , for any smooth vector fields X, Y in M ,*

$$(4.1) \quad s, t \in [0, 1]^2 \rightarrow \Omega_{Hc_t(s)}(X, Y)$$

is a smooth \mathfrak{g}_1 -valued map (for the \mathfrak{g}_1 -diffeology), and if M is simply connected, then the structure group G of P reduces to G_1 , and the connection θ also reduces.

Before giving the proof of this theorem, we need three lemmas, which are well-known results in finite dimensions (see e.g. [8]). Following [6], if $C_x^\infty([0, 1], M)$ is the set of smooth paths on M starting at x , we note

$$Pt : C_x^\infty([0, 1], M) \times [0, 1] \times \pi^{-1}(x) \rightarrow P$$

the parallel transport with respect to θ , which is a smooth map. Let us now describe the skeleton of the proof: the key tools for the definition of the local trivializations needed to reduce the principal bundle P is given in lemma 4.3, which is inspired by the computations in [6] in the case of a vanishing curvature and [10] in the general case. Lemmas 4.4 and 4.5 deal with local description of horizontal lifts. Finally, in the proof of Theorem 4.2, we define a family of local trivializations of P using Lemma 4.3, and check that it has the desired properties by Lemmas 4.4 and 4.5. Till the end of the proof of Theorem 4.2, we use the notations defined in the beginning of this paragraph.

Lemma 4.3. *Let $p \in P$. Let x be the basepoint of p . Let $\varphi : U \rightarrow M \in \mathcal{P}$ be a plot of the diffeology of M with star-shaped domain U , that we identify (for the sake of simplicity) with a star-shaped neighborhood of 0 in \mathbb{R}^n . Let $u \in U$ and $t \in [0, 1]$. We define $f(u, t) = tu \in U$.*

Let

$$\begin{aligned} \psi : U &\rightarrow P \\ u &\mapsto Pt(f(u, \cdot), 1, p). \end{aligned}$$

Let

$$\begin{aligned} \Psi : U \times G &\rightarrow \pi^{-1}(U) \\ (u, g) &\mapsto \psi(u).g, \end{aligned}$$

and

$$\tilde{\Psi} = \Psi \circ (Id_U \times \rho).$$

Then, ψ is a plot of the diffeology of P . Moreover, $\theta \circ D\psi$ is a smooth \mathfrak{g}_1 -valued form on U .

We need now to know how horizontal lifts of paths behave in the diffeology of P , and more precisely in with respect to the structure of G_1 .

Lemma 4.4. *We assume that U is convex.*

(i) Given a path $\alpha : [0, 1] \rightarrow U$ starting at x , if $H\alpha$ is its horizontal lift starting at p , we have $H\alpha(1) \in \Psi(U \times \rho(G_1))$, and there exists a smooth path $H\alpha_1 : [0, 1] \rightarrow U \times G_1$ such that $H\alpha = \tilde{\Psi} \circ H\alpha_1$.

(ii) Let $h : [0, 1]^2 \rightarrow U$ be an homotopy equivalence between two paths $h(0, \cdot)$ and $h(1, \cdot)$ starting at x and finishing in U . Let $Hh(0, \cdot)$ and $Hh(1, \cdot)$ be their horizontal lifts starting at p . Then, there is $g_1 \in G_1$ such that $\tilde{\Psi}^{-1}(Hh(0, 1)) = \tilde{\Psi}^{-1}(Hh(1, 1)).\rho(g_1)$.

Then, the following lemma will be useful when dealing with homotopy:

Lemma 4.5. *Let α and β be two paths on U . Let $q_\alpha \in \pi^{-1}(\alpha(0))$ and $q_\beta \in \pi^{-1}(\beta(0))$. Let $H\alpha$ and $H\beta$ be the horizontal lifts of α and β starting at q_α and q_β . We set $\tilde{\Psi}^{-1} \circ H\alpha = (\alpha, \gamma_\alpha)$ and $\Psi^{-1} \circ H\beta = (\beta, \gamma_\beta)$.*

Let $g = \gamma_\beta^{-1}(0).\gamma_\alpha(0)$, with $^{-1}$ as the inverse map in G . Then, for any $t \in [0, 1]$, there exists $g_1(t), g'_1 \in G_1$ such that $\gamma_\beta(t) = \gamma_\alpha(t).g_1^{-1}(t).g.g'_1(t)$. Moreover, the maps $t \mapsto g_1(t)$ and $t \mapsto g'_1(t)$ are smooth in G_1 .

Let us now give the proofs of the three lemmas, and then the proof of Theorem 4.2:

Proof of Lemma 4.3:

We already know that $\Psi : U \times G \rightarrow P$ is a smooth map, since Pt is smooth.

Let us calculate $D\Psi^{-1} \circ \theta \circ D\psi$. Let $c :]-\epsilon, \epsilon[\rightarrow U$ be a smooth path such that $c(0) = u \in U$. Let $h(t, s) = f(c(s), t)$. Let $\tilde{\theta}$ be the pull-back of θ by Ψ on U .

Then, following the proof of the claim of [6], theorem 39.2, with a connection with non vanishing curvature, see e.g. [10], we have:

$$\begin{aligned}
\partial_s(h^*\tilde{\theta})(\partial_t) &= \partial_t(h^*\tilde{\theta})(\partial_s) - d(h^*\tilde{\theta})(\partial_t, \partial_s) - (h^*\tilde{\theta})([\partial_t, \partial_s]) \\
&= \partial_t(h^*\tilde{\theta})(\partial_s) - d(h^*\tilde{\theta})(\partial_t, \partial_s) \\
&= \partial_t(h^*\tilde{\theta})(\partial_s) + ad_{(h^*\tilde{\theta})(\partial_t)}((h^*\tilde{\theta})(\partial_s)) - (\Psi^*\Omega)(h_*(\partial_s), h_*(\partial_t)) \\
&= \partial_t(h^*\tilde{\theta})(\partial_s) - (\Psi^*\Omega)(h_*(\partial_s), h_*(\partial_t)) \quad \text{since } (h^*\tilde{\theta})(\partial_t) = 0.
\end{aligned}$$

Thus,

$$\partial_s(\Psi \circ \psi \circ c) = (\partial_s c(s), \partial_s \tilde{\gamma}(1, s)),$$

remarking that $(u, e) = \Psi^{-1} \circ \psi(u)$. We now calculate $\partial_s \tilde{\gamma}(1, s)$,

$$\begin{aligned}
\partial_s \tilde{\gamma}(1, s) &= \int_0^1 \left(\partial_s(h^*\tilde{\theta})(\partial_t) \right) (t) dt \\
&= \int_0^1 \left(\partial_t(h^*\tilde{\theta})(\partial_s) - (\Psi^*\Omega)(h_*(\partial_s), h_*(\partial_t)) \right) (t) dt \\
&= (h^*\tilde{\theta})(\partial_s)(1, s) - \int_0^1 (\Psi^*\Omega)(h_*(\partial_s), h_*(\partial_t))(t) dt.
\end{aligned}$$

Finally, we have:

$$(4.2) \quad \theta(\partial_s(\psi \circ c)) = (h^*\tilde{\theta})(\partial_s h(1, s), \partial_s \gamma(1, s))$$

$$(4.3) \quad = \int_0^1 \left(\Omega(h_*(\partial_s), h_*(\partial_t))(t) \right) dt.$$

Since \mathfrak{g}_1 is complete, this integral exists and belongs to \mathfrak{g}_1 . \square

Proof of the Lemma 4.4:

(i) We have that $\theta(\partial_s(\psi \circ \alpha))$ is an integral on the curvature elements (see the proof of the last lemma). Looking at this result more precisely, reparametrizing equation (4.3), setting $c = \alpha$, we have that

$$(4.4) \quad \theta(\partial_s(h(s, t))) = \int_0^t \left(\Omega(h_*(\partial_s), h_*(\partial_t))(u) \right) du,$$

and hence that

$$\partial_t \left(\theta(\partial_s(h(s, t))) \right) = \Omega(h_*(\partial_s), h_*(\partial_t)).$$

Recall that $\rho^{-1} \circ \Omega(h_*(\partial_s), h_*(\partial_t))$ is smooth. Integrating this equality in G_1 instead of G , we get a path α_1 in $U \times G_1$. Then we consider the following differential equation, that defines $H\alpha_1$:

$$\begin{cases}
H\alpha_1(0) &= e \\
H\alpha_1(t) &= (\alpha(t), \gamma(t)) \in U \times G_1 \\
D\rho(\gamma(t)^{-1} \partial_t \gamma(t)) &= Ad_{(\rho \circ \gamma^{-1})(t)} \left(\theta \circ D\Psi \circ (Id \times \rho)(\partial_t \alpha_1)(t) \right),
\end{cases}$$

setting $H\alpha = \Psi \circ (Id_U \times \rho) \circ H\alpha_1$, we get (i).

(ii) comes easily from the continuity of the horizontal lift of paths, using the fact that U is contractible, and applying (i) to the path

$$(4.5) \quad c(t) = \begin{cases} Hh(0, 3t) & \text{if } t \in [0, 1/3] \\ Hh(3t - 1, 1) & \text{if } t \in [1/3, 2/3] \\ Hh(1, 3 - 3t) & \text{if } t \in [2/3, 1]. \end{cases}$$

□

Proof of Lemma 4.5: Reparametrizing Lemma 4.4, (i), we have that, for $t \in [0, 1]$, there exists $g_1(t), g'_1(t) \in G_1$ such that $\gamma_\alpha(t) = \gamma_\alpha(0).g_1(t)$ and $\gamma_\beta(t) = \gamma_\beta(0).g'_1(t)$. Then, we get $\gamma_\beta(t) = \gamma_\alpha(t).g_1^{-1}(t).g.g'_1(t)$. By equation (4.5), we have that the paths $t \mapsto g_1(t)$ and $t \mapsto g'_1(t)$ are smooth paths with values in G_1 . □

Proof of Theorem 4.2: Let $p_o \in P$, and let X be a family of paths γ_x in M , indexed by $x \in M$, starting at $x_0 = \pi(p_0)$ and ending at x . Let $X_1 = \bigcup_{x \in M} H\gamma_x(1)$ and let $P_1 = X_1.\rho(G_1)$. Let δ be an arbitrary path starting at x_0 , and let $x_1 = \delta(1)$. The path $\gamma_{x_1} \vee \delta^{-1}$ is null-homotopic since M is simply connected. So that, using Lemma 4.4, there exists $g_1 \in G_1$ such that $\gamma_{x_1} \vee \delta^{-1}(1) = p_0.\rho(g_1)$. So that $\delta(1) = \gamma_{x_1}(1).g_1^{-1}$ and $\pi^{-1}(x_1) = p_0.G_1$. Moreover, by Lemma 4.3, the maps $\varphi : U \rightarrow P$ are P_1 -valued. We define a smooth diffeology on P_1 which is generated by the push-forward diffeologies of the subsets of the type $Im(\varphi).G_1$, induced by the maps $(u, g_1) \in U \times G_1 \mapsto \varphi(u).G_1$. With this diffeology:

- the inclusion map $P_1 \rightarrow P$ is smooth
- the horizontal lift map $\alpha \mapsto H\alpha$ is a smooth map by trivial application of Lemma 4.5
- The connection θ restricts to a smooth \mathfrak{g}_1 -valued form by Lemma 4.3. This ends the proof of the reduction theorem, since G_1 -right invariant trivially. □

4.2. Ambrose-Singer theorem in the Frölicher setting. We can now state the announced Ambrose-Singer theorem. In this theorem, we review the key results of this paper:

Theorem 4.6. *Let P be a principal bundle of basis M with regular Frölicher structure group G with regular Lie algebra \mathfrak{g} . Let θ be a connection on P and L the associated path-lifting.*

a) *For each $p \in P$, the holonomy group \mathcal{H}_p^L is a diffeological subgroup of G , which does not depend on the choice of p up to conjugation.*

b) *There exists a second holonomy group $H^{\text{red}}, \mathcal{H} \subset H^{\text{red}}$, which is the smaller structure group for which there is a subbundle P' to which θ reduces. Its Lie algebra is spanned by the curvature elements, i.e. it is the smallest integrable Lie algebra which contains the curvature elements.*

c) *If G is a Lie group (in the classical sense) of type I or II in the terminology of Robart [14], there is closed Lie subgroup \bar{H}^{red} (in the classical sense) such that $H^{\text{red}} \subset \bar{H}^{\text{red}}$, whose Lie algebra is the closure in \mathfrak{g} of the Lie algebra of H^{red} , which is the smaller closed Lie subgroup of G among the structure groups of closed subbundles \bar{P}' of P to which θ reduces.*

Proof.

a) is proved in section 3

b) Let $P\mathcal{H}$ be the set of elements of P that are joint to p by a horizontal path. $P\mathcal{H}$ is obviously a principal bundle with structure group \mathcal{H} (or a “structure quantique” using the terminology of Souriau [13]). Notice that we do not assume here local trivializations on $P\mathcal{H}$. By Theorem 4.2, for each regular Frölicher Lie subgroup G_1 of G , with Lie algebra \mathfrak{g}_1 , if \mathfrak{g}_1 is regular, the connection θ reduces to the bundle

$$P\mathcal{H} \times_{\mathcal{H}} G_1 = (P\mathcal{H} \times G_1)/\mathcal{H}.$$

The family \mathfrak{G} of such Lie groups G_1 is not empty since $G \in \mathfrak{G}$, and it is obviously filtering for \supset . So that \mathfrak{G} has a minimal element H^{red} for \subset . By Theorem 1.18, the Lie algebra of H^{red} is the smaller regular Lie algebra which contains the curvature elements.

c) is a straightforward application of the arguments of [10].

5. APPENDIX: PROJECTIVE STRUCTURES

Let X and Λ be two sets, and $\{X_\alpha\}_{\alpha \in \Lambda}$ be a family of topological spaces indexed on Λ , with total order for inclusion. Let $\{f_\alpha : X \rightarrow X_\alpha\}_{\alpha \in \Lambda}$ be a family of maps such that, $\forall \alpha, \beta \in \Lambda^2$,

$$X_\alpha \subset X_\beta \Rightarrow (\exists i_{\beta, \alpha} X_\alpha \rightarrow X_\beta, \text{ continuous, such that } f_\beta = i_{\beta, \alpha} \circ f_\alpha).$$

The **projective topology** of X induced by the family of maps $\{f_\alpha\}_{\alpha \in \Lambda}$ is the weakest topology on X such that $\{f_\alpha\}_{\alpha \in \Lambda}$ is a family of continuous maps.

Example. Let $\{X_n\}_{n \in \mathbb{N}}$ be a countable family of topological vector spaces such that $\forall n \in \mathbb{N}$, $X_{n+1} \subset X_n$ with continuous inclusion i_n , and let $X = \bigcap_{n \in \mathbb{N}} X_n$. Then X can be given the projective topology of the family of inclusions and we denote this topological space as

$$X = \varprojlim X_n.$$

We call it **projective limit** of the family $\{X_n\}_{n \in \mathbb{N}}$.

We talk about projective limits of groups, vector spaces, algebras, manifolds, Lie groups, principal bundles, if the inclusions are morphisms of groups, vector spaces, algebras, manifolds, Lie groups, principal bundles.

Example. The Sobolev chain : (see e.g [12]) Let $\pi : E \rightarrow M$ be a finite rank vector bundle over a compact manifold without boundary M . Let $C^\infty(M, E)$ be the space of smooth sections of E , and $H^s(M, E)$ the space of H^s sections of E , equipped with their usual topologies. If $s < s'$, $C^\infty(M, E) \subset H^{s'}(M, E) \subset H^s(M, E)$ with continuous inclusion. If we denote by i_s the inclusion map $C^\infty(M, E) \subset H^s(M, E)$, we have that the topology of $C^\infty(M, E)$ is the projective topology of the family $\{i_s\}_{s \in \mathbb{R}}$, which is also the projective topology of the family $\{i_s\}_{s \in \mathbb{N}}$. Moreover, we have

$$C^\infty(M, E) = \varprojlim H^s(M, E).$$

Example. Families of equivalence relations. Let $\{R_\alpha\}_{\alpha \in \Lambda}$ be a family of equivalence relations on a set X , and let $X_\alpha = X/R_\alpha$. We denote by p_α the quotient projections. We assume that each X_α is equipped with a topological structure. Then X can be given a projective topology via the family $\{p_\alpha\}_{\alpha \in \Lambda}$. We have notice that the projective limit of a manifold is not in general a manifold modeled on the projective limit of the model spaces. This comes from the lack of assumption of atlas compatible with the projective limit. Another way to see this is to state that there is actually no inverse function theorem or implicit function theorem for the tame maps that we have defined, but only for restricted classes of maps, e.g. the maps with Nash-Moser estimates.

Projective limits of Lie groups have also the same singularity. One of the first known class of examples was developed by Omori, see e.g. [12], called generalized

Lie groups. These groups are the projective limit of an increasing sequence for \supset of Banach Lie groups, and they have an exponential map which is induced by the classical exponential map of Banach Lie groups.

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